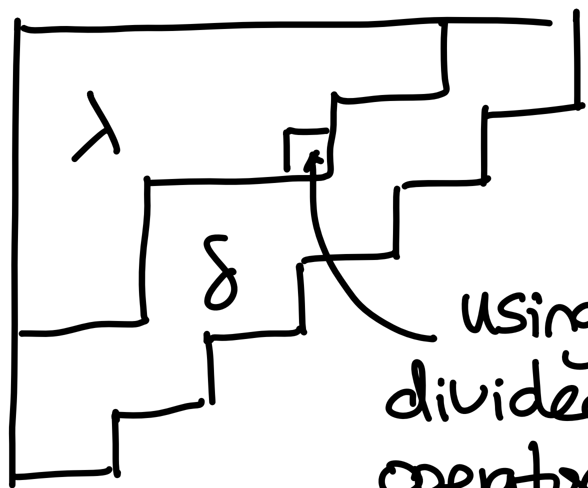
Last Time: any monomial $x^\lambda = x_1^{\lambda_1} \cdots x_n^{\lambda_n}$ s.t.

(1) $0 \le \lambda_i \le n-i$ for $i = 1, \dots, n$

(2) $\lambda_1 \ge \cdots \ge \lambda_n$

is a Certain Schubert Polynomial.



using the special divided difference operators we can remove cells.

"Special" divided difference

$$\partial_i \ x^\lambda \longmapsto x_1^{\lambda_1} \cdots x_i^{\lambda_i - 1} \cdots x_n^{\lambda_n}$$

where $\lambda_i = \lambda_{i+1} - 1$

Questions what are corresponding $w \in S_n$.

Lehmer code of perm. $w \in S_n$

$\mathrm{code}(w) = (c_1, \dots, c_n)$ s.t.

where $c_i = \# \{ j > i \mid w_j < w_i \}$

Ex $w = (1\ 5\ 2\ 3\ 4)$

$\mathrm{code} = (0\ 3\ 0\ 0\ 0)$

Lemma $w \longmapsto$ code is a bijection between S_n and $\{ (c_1, \dots, c_n) \mid 0 \le c_i \le n-i \}$

Lemma: TFAE for $w \in S_n$

(A) $\mathrm{code}(w) = (c_1, \dots, c_n)$ satisfies $c_1 \geq \cdots \geq c_n$

(B) w is 132-avoiding

no (i, j, k) with $i<j<k$ s.t. $w_i < w_k < w_j$

Ex $w = 2\ 1\ 4\ 5\ 3$

code $= 1\ 0\ 1\ 1\ 0$

Def 132-avoiding permutations are called dominant permutation.

Thm For a dominant permutation $w \in S_n$

$$\mathfrak{S}_w = x^{\mathrm{code}(w)}$$

Def (The Demazure operators D_i)
(aka isobaric divided differences)

for $i = 1, \dots, n-1$ and $f \in \mathbb{C}[x_1, \dots, x_n]$ def:

$$D_i : f \longmapsto \left(f - \frac{x_{i+1}}{x_i} s_i f\right) \Big/ \left(1 - \frac{x_{i+1}}{x_i}\right)$$

In other words $D_i = \partial_i (x_i f)$

Lemma: D_i satisfy

(1)'' $D_i^2 = D_i$

(2)'' $D_i D_{i+1} D_i = D_{i+1} D_i D_{i+1}$ $\quad i = 1, \dots, n-1$

(3)'' $D_i D_j = D_j D_i$ $\quad$ if $|i-j| \geq 2$.

0-Hecke relations

Hence $\forall\, w = s_{i_1} s_{i_2} \cdots s_{i_\ell}$ (reduced)

$$D_w := D_{i_1} \cdots D_{i_\ell}$$

Def Key polynomials (aka Demazure Char.)

$\lambda = (\lambda_1, \dots, \lambda_n)$ and $w \in S_n$

$$ch_{\lambda, w}(x_1, \dots, x_n) = D_w(x^\lambda)$$

Thm If $w = w_0 \in S_n$ then

$$ch_{\lambda, w_0}(x_1, \dots, x_n) = s_\lambda(x_1, \dots, x_n)$$

Proof: By defin. $S_\lambda = \partial_{w_0}(x^{\lambda+\delta})$

$$ch_{\lambda, w_0} = D_{w_0}(x^\lambda)$$

we need to show that

claim $\forall f \in \mathbb{C}[x_1, \dots, x_n]$

$$\partial_{w_0}(x^\delta f) = D_{w_0}(f)$$